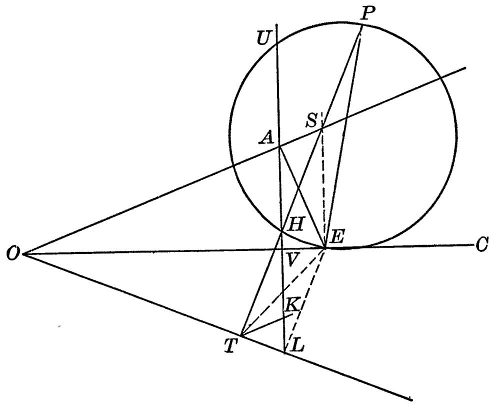
SOLUTION BY J. A. CAPARO, University of Notre Dame.

Let $\angle O$ be the given angle, P the given point, $OA = k$ half the required length.

Construction. Let OC bisect the given angle. Draw $AE \perp$ to OA and $AV \perp$ to OC, thus making $OL = OA$. On PE as a diameter describe a circle intersecting AL at U and H. Then PH and PU are the required lines.



Proof. Draw ES, ET and draw EL which will be $\perp$ to OL. The quadrilaterals $EHAS$ and $EHTL$ are inscriptible in circles since SAE and EHS; EHT and ELT are right angles.

Hence,

$$\angle SEA = \angle AHS = \angle THL = \angle TEL \quad \text{and} \quad \triangle ELT = \triangle EAS.$$

Then $ES = ET$, $HS = HT$ and $AS = TK = TL$, where TK is parallel to OA.

But, $OA + OL = 2k$ by construction. Hence,

$$OA + AS + OL - AS = 2k \quad \text{or} \quad OA + AS + OL - TL = 2k,$$

and finally,

$$OS + OT = 2k.$$

The line PU produced cuts from the given angle segments whose *difference* is $2k$. The proof is the same as the above.

An excellent solution was given by Professor F. L. GRIFFIN, of Reed College, the proof of which involves the calculus.

Using the same construction as in Professor Caparo's solution, he notes the fact that, after the point E is located, the rest is merely the well known construction of a tangent to a parabola from a given point. This parabola, which has OC as its axis, E as its focus, and AL as the tangent at its vertex, is the envelope of the whole family of lines cutting off on the sides of LO two segments whose sum is $2k$. Thus, the required line is merely that tangent to this parabola which passes through the given point P.

This point of view brings out the discussion that there is no solution, one solution, or two solutions, according as the given point P lies within, on, or without the parabola. If P lies outside the given angle, as in the figure above, then one of the lines, as PU, will cut one side of the angle produced through the vertex, but the sum will still be $2k$ if we call this segment negative. Professor Griffin also gives a solution involving analytic geometry but not the calculus. Others may be interested to work out both of these solutions. EDITORS.

Also solved by A. H. HOLMES, PAUL CAPRON, J. W. CLAWSON, and N. P. PANDYA.

468. Proposed by ELMER SCHUYLER, Brooklyn, New York.

Given two circles and a straight line, to draw a circle tangent to the line and coaxial with the two given circles.

Solution by Nathan Altshiller, University of Colorado.

Let r be the radical axis of the two given circles γ_1, γ_2, and P the point of intersection of r with the given line, l. The tangents drawn from any point of r, and in particular from P, to all the circles of the coaxial system determined by γ_1 and γ_2, are all of equal length, when measured from P to the respective points of contact. On the other hand, the centers of all the circles coaxial with the two given circles, lie on the line of centers c of these two circles.

The above suggests the following solution of the problem:

From the point P draw a tangent to one of the given circles. Let S denote the point of contact. On l lay off two segments PT and PT' such that $PT = PT' = PS$. The perpendiculars to l erected at T and T' will meet the line of centers c in the centers C and C' of the two circles satisfying the conditions of the problem, the respective radii being the segments CT and $C'T'$.

This construction is applicable whether the given circles are tangent to each other, or cut each other in real, or imaginary points.

The problem has, in general, two real solutions.

An interesting special case arises when the line l is parallel to the radical axis r, and hence perpendicular to the line of centers c. Since the center of the required circle is to be on c, the necessary and sufficient condition for it to be tangent to l is, in the present case, that the circle shall pass through the point of intersection of l with c. We are thus led to the problem:

Given two circles and a point, to draw a circle passing through the given point and coaxial with the two given circles.

This problem may be solved as follows: From an arbitrary point P of the radical axis r draw a tangent PR to one of the given circles, touching the circle at the point R. On the line PQ joining P to the given point Q find the point Q' such that $PQ \cdot PQ' = \overline{PR}^2$, Q and Q' being on the same side of the radical axis. The point Q' belongs to the required circle, which is now readily constructed.

This construction is valid for any point Q in the plane, and whatever the relative position of the two circles with respect to each other may be.

The problem has, in general, one real solution.

Also solved by C. N. Schmall, Geo. W. Hartwell, Frank Irwin, Herbert N. Carleton, J. W. Clawson, and N. P. Pandya.

469. Proposed by J. ALEXANDER CLARKE, West Philadelphia High School.

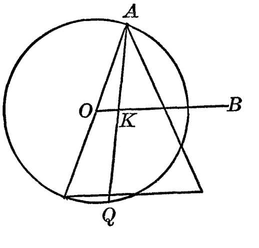
If in an isosceles triangle, a circle is described on one side as diameter, and a line is drawn through the mid-point of the side parallel to the base, the circle and the parallel will intercept on the trisector of the angle at the vertex a segment equal to the radius of the circle. Show how this can be used to trisect any angle.

Solution by J. W. Clawson, Collegeville, Pa.

Let A be the vertex of the isosceles triangle. Bisect either arm at O. Draw the circle, center O, radius OA. Draw a line, OB, from O parallel to the base. Then if a line is supposed drawn to trisect the angle A, cutting the line OB at K and cutting the circle again at Q, we are to prove that $KQ = R$, the radius of the circle.

Now

$$AQ = 2R\cos\frac{A}{3}.$$



Also

$$AK = R\frac{\sin(90^\circ - A/2)}{\sin(90^\circ - A/2 + A/3)} = R\frac{\cos\dfrac{A}{2}}{\cos\dfrac{A}{6}}.$$

Hence

$$KQ = R\left[2\cos\frac{A}{3} - \cos\frac{A}{2}\sec\frac{A}{6}\right] = R\left[2\cos\frac{A}{3} - 4\cos^2\frac{A}{6} + 3\right]$$

$$= R\left[2\cos\frac{A}{3} - 2\left(1 + \cos\frac{A}{3}\right) + 3\right] = R.$$